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INTERACTION BETWEEN AN INFINITESIMAL GLIDE DISLOCATION LOOP COPLANAR WITH A PENNY-SHAPED CRACK

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Abstract—This analysis develops a Green's function for a point shear dislocation coplanar with a penny-shaped crack. The solution procedure consists of using anti-symmetry conditions which allow the reduction to an asymmetric mixed boundary value problem for a half space. The mixed boundary conditions between the shear stresses and displacements on the surface are formulated in terms of a single pair and two coupled pairs of dual integral equations which have known solutions. Closed form expressions are found for the shear displacement discontinuity inside the crack as well as for the shear stress on the crack–dislocation plane. The dislocation interaction solution is then used to formulate new integral equations for shear loading of multiple coplanar cracks or cracks with nonuniform crack fronts.

1. INTRODUCTION

Recent effort in the area of three-dimensional fracture mechanics has been aimed at the development of a method for the analysis of the nonuniform advancement of a crack front. A crack front may advance in a nonuniform manner due to several factors such as the nature of the applied loading or interactions of the crack with regions of damage, heterogeneity or transformations. Typical examples of these crack interactions are the formation of cleavage fractures or microcracks in a brittle material ahead of a growing crack, heterogeneities placed in a matrix to improve fracture resistance and strength, or the transformation toughening phenomenon of second phase particles in ceramic materials. One method of analysis for these three-dimensional fracture problems is the formulation of hyper-singular integral equations, by distributing dislocations over the entire cracked domain, which can be solved numerically [see, for example, Murakami and Nemat-Nasser (1982, 1983), Lee *et al.* (1987) and Hanson *et al.* (1989)]. A straightforward numerical solution of boundary integral equations over the crack domain requires discretizing the crack into elements over which the crack opening displacement is given appropriate representations. However, when the cracked domain is large and there are high amplitude variations in the crack front curvature, a large number of elements are often needed to obtain results of acceptable accuracy. This necessitates the need for large computer storage and computational time. A first order perturbation method, which overcomes this difficulty, has been developed by several investigators to analyse nonuniform crack fronts.

Apparently the first such formulation was given by Panasyuk (1962) [see Appendix B of Gao and Rice (1987a)] who developed a first order perturbation solution for the special case of a planar tensile crack of near circular geometry. A general first order theory was constructed by Rice (1985a) for the variation in the elastic field in a cracked body caused by perturbation in the crack front position. His analysis considered mode I loading of a half plane crack. Gao and Rice (1986) extended this work to the shear mode case. The first order analysis was applied to the planar circular internal crack under mode I loading by Gao and Rice (1987a) and Gao and Rice (1987b) considered the external circular crack. The nearly circular internal crack under shear loading was analysed by Gao (1988). The first order theory developed relies on the knowledge of the weight function (Bueckner, 1970, 1973; Rice, 1972) which gives the stress intensity factor distribution around a crack front caused by an arbitrarily located point body force. Certain crack face weight functions (a limiting case for a generally located point force) have been known for some time, however the general weight functions for the half plane and circular cracks have only been recently determined (Rice, 1985a; Bueckner, 1987; Gao, 1989a).

Although a first order theory avoids the heavy numerical computations, it is quantitatively limited. To overcome this difficulty, Rice (1987) has proposed two alternative procedures. One method is a series of integrations of the first order theory. This amounts to numerically determining a new weight function for the perturbed crack geometry and sequentially applying the first order theory. This method was recently used by Bower and Ortiz (1990). Secondly, he proposed the solution of integral equations over a reduced crack domain by distributing dislocations between the wavy crack boundary and that of the reference (half plane or circular) crack. This solution procedure was recently used successfully by Fares (1989) in analysing a half plane crack with a wavy crack front and crack trapping by arrays of obstacles.

This second procedure requires the determination of the interaction between the reference crack and a point dislocation. As demonstrated by Rice (1985a), the weight function provides information not only on the stress intensity factor for a point body force and on the stress intensity factor around a perturbed crack front, but it can also be used to determine the interactions between sources of internal stress (such as dislocations) and crack tips. His analysis showed that these interactions could be given as double integrals of products of weight functions over the entire crack history. This property was used by Rice (1985b) to analyse the interactions between a half plane crack and transformation strains or dislocations. The double integral was evaluated in closed form to give the stress field ahead of a half plane crack when a mode I point dislocation was on the crack plane ahead of the crack. This is the point dislocation Green's function used in the numerical analysis by Fares (1989). Anderson and Rice (1987) also utilized this solution to give a detailed consideration of the stress field and self energy of semicircular and rectangular coplanar dislocation loops emanating from a half plane crack under mode I loading. The shear mode case for the half plane was studied by Gao (1989b) and Gao and Rice (1989). The closed form shear stress ahead of a half plane crack caused by a shear mode dislocation coplanar with the crack was given by Gao and Rice (1989). Karihaloo and Huang (1989) have also used this method to study the interactions between a half plane crack and regions of shear transformation strains.

The double integrals of products of weight functions over the crack history, required to determine the interactions, are formidable and have only been evaluated in closed form for the two special cases of the half plane crack noted above. Recently, in applying the distributed dislocation idea of Rice (1985b, 1987) to the large amplitude variation of a finite size crack front, Hanson (1990, 1991) determined in closed form the interaction between a point prismatic dislocation coplanar with a circular internal or external crack for a transversely isotropic material. The solution was found by using well known methods from potential theory rather than considering double integrals of Bueckner's (1987) and Gao's (1989a) weight functions for the circular crack. The dislocation interaction solution was used to derive Fabrikant's (1987a) nonsingular integral equations for multiple coplanar circular cracks as well as to derive new singular integral equations over a reduced crack domain for use in analysing multiple coplanar cracks and planar cracks with large amplitude variations in the crack front curvature.

In the present analysis, the solution of Hanson (1990) is extended to the shear mode case. Again, methods from potential theory are utilized and an extension is made to the method developed by Westman (1965) to formulate the problem in terms of a single pair and coupled pairs of dual integral equations which have known solutions. Having the solution to the integral equations, the shear displacement discontinuities inside the crack and the shear stress on the crack dislocation plane are found in closed form. Although the coplanar shear dislocation is certainly not the most interesting case from the standpoint of crack tip dislocation emission as studied by Gao and Rice (1989), the solution obtained is valuable in considering multiple coplanar cracks or planar cracks with large variations in the crack front curvature subjected to shear loading. New singular integral equations over a reduced crack domain are obtained which can be used to accurately analyse a finite size planar crack with a wavy crack front or multiple coplanar cracks subjected to shear loading.

2. FORMULATION

The full space dislocation interaction problem studied in this analysis is reduced by anti-symmetry conditions to a mixed boundary value problem for a half space. A convenient starting point for determining a solution is the potential function representation for displacements and stresses in a half space $z > 0$ given by Muki (1960). The cylindrical coordinates (r, θ, z) are used and the displacements u, v and w are in the r, θ and z directions, respectively. In the sequel μ is the shear modulus and ν is Poisson's ratio. The displacements and stresses are given in terms of a biharmonic function Φ and a harmonic function ψ. The interested reader is referred to Muki (1960) for the explicit relations.

For the cylindrical geometry considered here, appropriate representations for the functions Φ and ψ in the half space $z > 0$ are given as

$$\psi(r,\theta,z) = \sum_0^\infty \int_0^\infty \xi D_n^c(\xi)\,\mathrm{e}^{-\xi z} J_n(\xi r)\,\mathrm{d}\xi \cos(n\theta) + \sum_1^\infty \int_0^\infty \xi D_n^s(\xi)\,\mathrm{e}^{-\xi z} J_n(\xi r)\,\mathrm{d}\xi \sin(n\theta), \quad (1)$$

$$\Phi(r,\theta,z) = \sum_0^\infty \int_0^\infty \xi[E_n^c(\xi)+zF_n^c(\xi)]\,\mathrm{e}^{-\xi z} J_n(\xi r)\,\mathrm{d}\xi \cos(n\theta) + \sum_1^\infty \int_0^\infty \xi[E_n^s(\xi)+zF_n^s(\xi)]\,\mathrm{e}^{-\xi z} J_n(\xi r)\,\mathrm{d}\xi \sin(n\theta). \quad (2)$$

The quantities which will be of interest are on the surface $z = 0$ of the half space and consist of the shear displacements $u(r, \theta, 0)$ and $v(r, \theta, 0)$ as well as the stresses $\sigma_{zz}(r, \theta, 0)$, $\tau_{zr}(r, \theta, 0)$ and $\tau_{\theta z}(r, \theta, 0)$. Considering the normal stress first, it is easy to show that

$$\sigma_{zz}(r,\theta,0) = 2\mu \sum_0^\infty \int_0^\infty \xi^3[(1-2\nu)F_n^c(\xi)+\xi E_n^c(\xi)]J_n(\xi r)\,\mathrm{d}\xi \cos(n\theta) + 2\mu \sum_1^\infty \int_0^\infty \xi^3[(1-2\nu)F_n^s(\xi)+\xi E_n^s(\xi)]J_n(\xi r)\,\mathrm{d}\xi \sin(n\theta). \quad (3)$$

In the following analysis, a solution with vanishing normal stress on the surface is required. This condition is satisfied by the relations

$$(1-2\nu)F_n^c(\xi) = -\xi E_n^c(\xi), \quad (1-2\nu)F_n^s(\xi) = -\xi E_n^s(\xi). \quad (4)$$

In order to match boundary conditions on the surface, the shear stresses and displacements are separated as follows. Taking the cosine expansion of Φ and the sine expansion of ψ and using eqn (4) results in

$$u(r,\theta,0) = 2(1-\nu)\int_0^\infty \xi^2 F_0^c(\xi) J_1(\xi r)\,\mathrm{d}\xi - \sum_1^\infty \int_0^\infty \xi^2[(1-\nu)F_n^c(\xi) - D_n^s(\xi)] \times J_{n-1}(\xi r)\,\mathrm{d}\xi \cos(n\theta) + \sum_1^\infty \int_0^\infty \xi^2[(1-\nu)F_n^c(\xi)+D_n^s(\xi)]J_{n+1}(\xi r)\,\mathrm{d}\xi \cos(n\theta), \quad (5)$$

$$v(r,\theta,0) = \sum_1^\infty \int_0^\infty \xi^2[(1-\nu)F_n^c(\xi) - D_n^s(\xi)]J_{n-1}(\xi r)\,\mathrm{d}\xi \sin(n\theta) + \sum_1^\infty \int_0^\infty \xi^2[(1-\nu)F_n^c(\xi)+D_n^s(\xi)]J_{n+1}(\xi r)\,\mathrm{d}\xi \sin(n\theta), \quad (6)$$

$$\tau_{zr}(r,\theta,0) = \mu\left\{-2\int_0^\infty \xi^3 F_0^c(\xi) J_1(\xi r)\,\mathrm{d}\xi + \sum_1^\infty \int_0^\infty \xi^3 [F_n^c(\xi) - D_n^s(\xi)] \right.$$

$$\left. \times J_{n-1}(\xi r)\,\mathrm{d}\xi \cos(n\theta) - \sum_1^\infty \int_0^\infty \xi^3 [F_n^c(\xi) + D_n^s(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \cos(n\theta)\right\}. \quad (7)$$

$$\tau_{\theta z}(r,\theta,0) = \mu\left\{-\sum_1^\infty \int_0^\infty \xi^3 [F_n^c(\xi) - D_n^s(\xi)] J_{n-1}(\xi r)\,\mathrm{d}\xi \sin(n\theta) \right.$$

$$\left. - \sum_1^\infty \int_0^\infty \xi^3 [F_n^c(\xi) + D_n^s(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \sin(n\theta)\right\}. \quad (8)$$

Taking the sine expansion of Φ and the cosine expansion of ψ and using eqn (4) gives

$$u(r,\theta,0) = -\sum_1^\infty \int_0^\infty \xi^2 [(1-\nu) F_n^s(\xi) + D_n^c(\xi)] J_{n-1}(\xi r)\,\mathrm{d}\xi \sin(n\theta)$$

$$+ \sum_1^\infty \int_0^\infty \xi^2 [(1-\nu) F_n^s(\xi) - D_n^c(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \sin(n\theta), \quad (9)$$

$$v(r,\theta,0) = 2\int_0^\infty \xi^2 D_0^c(\xi) J_1(\xi r)\,\mathrm{d}\xi - \sum_1^\infty \int_0^\infty \xi^2 [(1-\nu) F_n^s(\xi) + D_n^c(\xi)]$$

$$\times J_{n-1}(\xi r)\,\mathrm{d}\xi \cos(n\theta) - \sum_1^\infty \int_0^\infty \xi^2 [(1-\nu) F_n^s(\xi) - D_n^c(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \cos(n\theta), \quad (10)$$

$$\tau_{zr}(r,\theta,0) = \mu\left\{\sum_1^\infty \int_0^\infty \xi^3 [F_n^s(\xi) + D_n^c(\xi)] J_{n-1}(\xi r)\,\mathrm{d}\xi \sin(n\theta) \right.$$

$$\left. - \sum_1^\infty \int_0^\infty \xi^3 [F_n^s(\xi) - D_n^c(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \sin(n\theta)\right\}, \quad (11)$$

$$\tau_{\theta z}(r,\theta,0) = \mu\left\{-2\int_0^\infty \xi^3 D_0^c(\xi) J_1(\xi r)\,\mathrm{d}\xi + \sum_1^\infty \int_0^\infty \xi^3 [F_n^s(\xi) + D_n^c(\xi)] \right.$$

$$\left. \times J_{n-1}(\xi r)\,\mathrm{d}\xi \cos(n\theta) + \sum_1^\infty \int_0^\infty \xi^3 [F_n^s(\xi) - D_n^c(\xi)] J_{n+1}(\xi r)\,\mathrm{d}\xi \cos(n\theta)\right\}. \quad (12)$$

3. POINT DISLOCATION IN A FULL SPACE

The solution for an arbitrary shear dislocation in a full space is determined first. This solution is well known but certain results are necessary for the interaction problem analysed subsequently. Consider the point dislocation to be on the plane $z = 0$ of the full space $-\infty < z < \infty$, $0 < r < \infty$ and $0 < \theta < 2\pi$ located at the arbitrary point (r_0, θ_0). Let the dislocation have an r-directed opening of magnitude b_r and a θ-directed opening of b_θ. The plane $z = 0$ is then one of anti-symmetry on which the normal stress vanishes. The shear displacements u and v are zero everywhere on the surface except at the dislocation and may be written as

$$\begin{Bmatrix} u(r,\theta,0) \\ v(r,\theta,0) \end{Bmatrix} = \begin{Bmatrix} b_r \\ b_\theta \end{Bmatrix} \frac{1}{2r}\delta(r-r_0)\delta(\theta-\theta_0). \tag{13}$$

These expressions may be expanded in a Fourier series to give

$$\begin{Bmatrix} u(r,\theta,0) \\ v(r,\theta,0) \end{Bmatrix} = \begin{Bmatrix} b_r \\ b_\theta \end{Bmatrix} \frac{1}{2\pi r}\delta(r-r_0)\left\{\frac{1}{2}+\sum_1^\infty \cos\,[n(\theta-\theta_0)]\right\}. \tag{14}$$

The unknown functions $F^c_n(\xi)$, $F^s_n(\xi)$, $D^c_n(\xi)$ and $D^s_n(\xi)$ are determined by matching the above boundary conditions. From eqns (5) and (6), one set of conditions becomes

$$2(1-\nu)\int_0^\infty \xi^2 F^c_0(\xi)J_1(\xi r)\,\mathrm{d}\xi = \frac{b_r}{4\pi r}\delta(r-r_0), \tag{15}$$

$$\int_0^\infty \xi^2[(1-\nu)F^s_n(\xi)+D^s_n(\xi)]J_{n+1}(\xi r)\,\mathrm{d}\xi = \frac{1}{4\pi r}\delta(r-r_0)[b_r\cos(n\theta_0)+b_\theta\sin(n\theta_0)], \tag{16}$$

$$\int_0^\infty \xi^2[(1-\nu)F^s_n(\xi)-D^s_n(\xi)]J_{n-1}(\xi r)\,\mathrm{d}\xi = \frac{-1}{4\pi r}\delta(r-r_0)[b_r\cos(n\theta_0)-b_\theta\sin(n\theta_0)], \tag{17}$$

which have the solution

$$2\pi(\kappa+1)\xi F^c_0(\xi) = b_r J_1(\xi r_0), \tag{18}$$

$$\begin{aligned}2\pi(\kappa+1)\xi F^s_n(\xi) = {} & b_r\cos(n\theta_0)[J_{n+1}(\xi r_0)-J_{n-1}(\xi r_0)] \\ & +b_\theta\sin(n\theta_0)[J_{n+1}(\xi r_0)+J_{n-1}(\xi r_0)],\end{aligned} \tag{19}$$

$$8\pi\xi D^s_n(\xi) = b_r\cos(n\theta_0)[J_{n+1}(\xi r_0)+J_{n-1}(\xi r_0)]+b_\theta\sin(n\theta_0)[J_{n+1}(\xi r_0)-J_{n-1}(\xi r_0)], \tag{20}$$

where $\kappa = 3-4\nu$. From eqns (9) and (10) the additional conditions are

$$2\int_0^\infty \xi^2 D^c_0(\xi)J_1(\xi r)\,\mathrm{d}\xi = \frac{b_\theta}{4\pi r}\delta(r-r_0), \tag{21}$$

$$\int_0^\infty \xi^2[(1-\nu)F^c_n(\xi)+D^c_n(\xi)]J_{n-1}(\xi r)\,\mathrm{d}\xi = \frac{-1}{4\pi r}\delta(r-r_0)[b_r\sin(n\theta_0)+b_\theta\cos(n\theta_0)], \tag{22}$$

$$\int_0^\infty \xi^2[(1-\nu)F^c_n(\xi)-D^c_n(\xi)]J_{n+1}(\xi r)\,\mathrm{d}\xi = \frac{1}{4\pi r}\delta(r-r_0)[b_r\sin(n\theta_0)-b_\theta\cos(n\theta_0)], \tag{23}$$

which have the solution

$$8\pi\xi D^c_0(\xi) = b_\theta J_1(\xi r_0), \tag{24}$$

$$\begin{aligned}2\pi(\kappa+1)\xi F^c_n(\xi) = {} & b_r\sin(n\theta_0)[J_{n+1}(\xi r_0)-J_{n-1}(\xi r_0)] \\ & -b_\theta\cos(n\theta_0)[J_{n+1}(\xi r_0)+J_{n-1}(\xi r_0)],\end{aligned} \tag{25}$$

$$-8\pi\xi D^c_n(\xi) = b_r\sin(n\theta_0)[J_{n+1}(\xi r_0)+J_{n-1}(\xi r_0)]-b_\theta\cos(n\theta_0)[J_{n+1}(\xi r_0)-J_{n-1}(\xi r_0)]. \tag{26}$$

The shear stress on the plane of the dislocation is the quantity of interest and can be found by substituting eqns (18)–(20) into (7)–(8) and (24)–(26) into (11)–(12) and superposing these expressions to obtain the total shear stress. These expressions may be evaluated by first writing the Bessel functions as

$$J_{n-1}(\xi r)-J_{n+1}(\xi r)=\frac{2}{\xi}\frac{\partial}{\partial r}J_n(\xi r),\quad J_{n-1}(\xi r)+J_{n+1}(\xi r)=\frac{2n}{\xi r}J_n(\xi r),\tag{27}$$

with similar relations holding for $J_n(\xi r_0)$. After applying eqn (27), the sums may be evaluated first. Here use is made of Neumann's addition theorem (Watson, 1980)

$$\sum_{1}^{\infty}\cos\left[n(\theta-\theta_0)\right]J_n(\xi r_0)J_n(\xi r)=\tfrac{1}{2}[J_0(\xi R)-J_0(\xi r_0)J_0(\xi r)],\tag{28}$$

with $R^2=r^2+r_0^2-2rr_0\cos(\theta-\theta_0)$. The additional sums can be evaluated by differentiating the above expression with respect to θ. Evaluating the integrals next as follows

$$\int_0^{\infty}J_0(\xi R)\,\mathrm{d}\xi=\frac{1}{R},\quad \int_0^{\infty}\xi J_1(\xi R)\,\mathrm{d}\xi=\frac{1}{R^2},\quad \int_0^{\infty}\xi^2 J_0(\xi R)\,\mathrm{d}\xi=-\frac{1}{R^3},\tag{29}$$

and then performing the required differentiations from eqn (27) allows closed form expressions to be obtained. The shear stresses may be reduced to the form

$$\begin{aligned}\frac{4\pi(\kappa+1)}{\mu}\tau_{zr}(r,\theta,0)=&\frac{2(\kappa-1)}{R^3}[b_r\cos(\theta-\theta_0)+b_\theta\sin(\theta-\theta_0)]\\&+\frac{3(3-\kappa)}{R^5}b_r\{(r^2+r_0^2)\cos(\theta-\theta_0)-rr_0[1+\cos^2(\theta-\theta_0)]\}\\&+\frac{3(3-\kappa)}{R^5}b_\theta[r^2\sin(\theta-\theta_0)-rr_0\cos(\theta-\theta_0)\sin(\theta-\theta_0)],\end{aligned}\tag{30}$$

$$\begin{aligned}\frac{4\pi(\kappa+1)}{\mu}\tau_{\theta z}(r,\theta,0)=&\frac{2(\kappa-1)}{R^3}[b_\theta\cos(\theta-\theta_0)-b_r\sin(\theta-\theta_0)]\\&+\frac{3(3-\kappa)}{R^5}b_r[-r_0^2\sin(\theta-\theta_0)+rr_0\cos(\theta-\theta_0)\sin(\theta-\theta_0)]\\&+\frac{3(3-\kappa)}{R^5}b_\theta rr_0\sin^2(\theta-\theta_0).\end{aligned}\tag{31}$$

It may be shown, by converting these expressions to Cartesian components, that the present results agree with the known expressions [see, for example, Lee *et al.* (1987)].

4. DISLOCATION INTERACTION WITH A PENNY-SHAPED CRACK

Consideration is now given to the interaction between a point shear dislocation and a penny-shaped crack. As shown in Fig. 1, the crack occupies the region $z = 0$, $r < a$ while the dislocation is at the point $(r_0, \theta_0, 0)$ with $r_0 > a$. The dislocation has displacement discontinuities of magnitude b_r and b_θ in the r and θ directions respectively. From the antisymmetry conditions, the problem is again reduced to one for a half space with zero normal stress applied on the surface. The shear boundary conditions on the stresses and

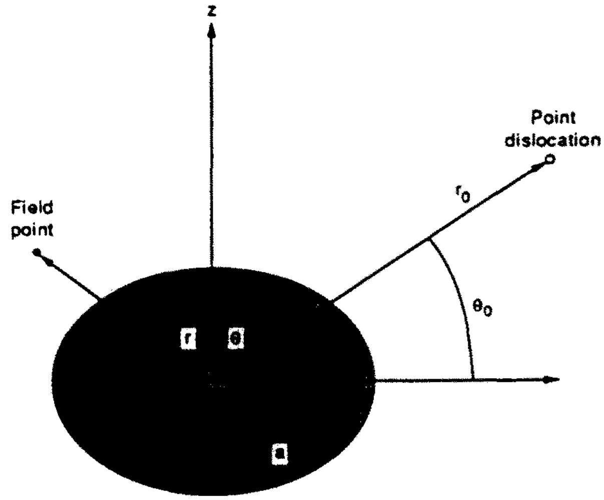

Fig. 1. Point shear dislocation coplanar with a penny-shaped crack.

displacements can now be given as

$$\tau_{\theta z}(r,\theta,0) = 0, \quad \tau_{rz}(r,\theta,0) = 0, \quad r < a, \tag{32}$$

$$\begin{Bmatrix} u(r,\theta,0) \\ v(r,\theta,0) \end{Bmatrix} = \begin{Bmatrix} b_r \\ b_\theta \end{Bmatrix} \frac{1}{2r}\delta(r-r_0)\delta(\theta-\theta_0), \quad r > a. \tag{33}$$

As with the full space solution, the boundary conditions are met in two parts. The first step is to consider the cosine expansion of Φ and the sine expansion of ψ [thus utilizing eqns (5)–(8)]. The transform functions are first redefined as

$$\xi^2 F_0^c(\xi) = \psi_0(\xi), \qquad 2\xi^2 F_n^c(\xi) = \psi_2(\xi) - \frac{\nu}{2-\nu}\psi_1(\xi), \tag{34, 35}$$

$$2\xi^2 D_n^s(\xi) = \psi_2(\xi) + \frac{\nu}{2-\nu}\psi_1(\xi). \tag{36}$$

Making the following change of variables

$$r = a\rho, \quad r_0 = a\rho_0, \quad \xi = y/a, \quad n = m+1, \quad b = \frac{(2-\nu)^2}{\nu^2}, \tag{37}$$

and with the substitutions

$$\psi_0(y/a) = \bar{\psi}_0(y), \quad \psi_1(y/a) = \bar{\psi}_1(y), \quad \psi_2(y/a) = \bar{\psi}_2(y), \tag{38}$$

the contribution of these terms to the boundary conditions in eqns (32)–(33) can be written as

$$\int_0^\infty y\bar{\psi}_0(y)J_1(y\rho)\,dy = 0, \quad \rho < 1, \qquad \int_0^\infty \bar{\psi}_0(y)J_1(y\rho)\,dy = f(\rho), \quad \rho > 1, \tag{39, 40}$$

$$\int_0^\infty y\bar{\psi}_2(y)J_{m+2}(y\rho)\,dy = 0, \quad \rho < 1, \qquad \int_0^\infty y\bar{\psi}_1(y)J_m(y\rho)\,dy = 0, \quad \rho < 1, \tag{41, 42}$$

$$\frac{2}{\rho}\int_0^\infty [\bar{\psi}_1(y)+b\bar{\psi}_2(y)]J_{m+2}(y\rho)\,\mathrm{d}y = j(\rho), \quad \rho > 1, \tag{43}$$

$$\frac{2}{\rho}\int_0^\infty [\bar{\psi}_1(y)+\bar{\psi}_2(y)]J_m(y\rho)\,\mathrm{d}y = h(\rho), \quad \rho > 1. \tag{44}$$

Here the functions $f(\rho)$, $j(\rho)$ and $h(\rho)$ are defined as

$$f(\rho) = \frac{b_r}{2\pi(\kappa+1)\rho}\delta(a[\rho-\rho_0]), \tag{45}$$

$$j(\rho) = \frac{2-v}{\pi v^2\rho^2}\delta(a[\rho-\rho_0])[b_r\cos(n\theta_0)+b_\theta\sin(n\theta_0)], \tag{46}$$

$$h(\rho) = \frac{1}{\pi v\rho^2}\delta(a[\rho-\rho_0])[b_r\cos(n\theta_0)-b_\theta\sin(n\theta_0)]. \tag{47}$$

The solution to the dual integral equations (39) and (40) is easily found as (Sneddon, 1966)

$$\bar{\psi}_0(y) = y^{1/2}\int_1^\infty \phi(t)J_{3/2}(yt)\,\mathrm{d}t, \quad \phi(t) = -\frac{\sqrt{2}}{\sqrt{\pi}}t^{3/2}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{f(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-t^2}}. \tag{48}$$

Substituting for $f(\rho)$ and using eqns (34), (37) and (38) results in

$$F_0^\infty(\xi) = -\frac{1}{\sqrt{2\pi}}\frac{b_r}{\pi(\kappa+1)}\xi^{-3/2}r_0^{-1}\int_a^{r_0}\frac{\beta^{5/2}}{(r_0^2-\beta^2)^{3/2}}J_{3/2}(\xi\beta)\,\mathrm{d}\beta. \tag{49}$$

The integral is divergent at the upper limit. This is associated with the dislocation in a full space ($a = 0$) as discussed by Hanson (1990). Evaluating the divergent integral as follows

$$\int_0^{r_0}\frac{\beta^{5/2}J_{3/2}(\xi\beta)\,\mathrm{d}\beta}{(r_0^2-\beta^2)^{3/2}} = -\frac{\sqrt{\pi\xi}}{\sqrt{2}}r_0J_1(\xi r_0), \tag{50}$$

gives the result

$$F_0^\infty(\xi) = \frac{b_r}{2\pi(\kappa+1)\xi}\left[J_1(\xi r_0)+\frac{\sqrt{2}}{\sqrt{\pi\xi}}r_0^{-1}\int_0^a\frac{\beta^{5/2}}{(r_0^2-\beta^2)^{3/2}}J_{3/2}(\xi\beta)\,\mathrm{d}\beta\right]. \tag{51}$$

From eqn (18), it is easy to verify that the first term of eqn (51) represents the full space contribution. The integral represents the interaction which vanishes for $a \to 0$.

Equations (41)–(44) are coupled dual integral equations to determine the unknown function $\bar{\psi}_1(y)$ and $\bar{\psi}_2(y)$. The solution is given by Keer (1968) to be

$$\bar{\psi}_1(y) = -b\frac{\sqrt{y}}{\sqrt{2}}\int_1^\infty \beta^{3/2}J_{m+1/2}(y\beta)h_2(\beta)\,\mathrm{d}\beta, \tag{52}$$

$$\bar{\psi}_2(y) = -\frac{1}{b}\bar{\psi}_1(y)+\frac{\sqrt{y}}{\sqrt{2}}\int_1^\infty \beta^{3/2}J_{m+5/2}(y\beta)h_3(\beta)\,\mathrm{d}\beta, \tag{53}$$

$$h_3(\beta) = -\frac{\beta^{m+1}}{b\sqrt{\pi}}\frac{d}{d\beta}\int_\beta^\infty \frac{\rho^{-m}j(\rho)\,d\rho}{\sqrt{\rho^2-\beta^2}}, \qquad h_2(\beta) = \frac{\beta^{m-1}}{b(1-b)\sqrt{\pi}}\frac{d}{d\beta}\int_\beta^\infty \frac{\rho^{-m}g(\rho)\,d\rho}{\sqrt{\rho^2-\beta^2}}, \tag{54, 55}$$

'here

$$g(\rho) = -b\rho^2 h(\rho) + (2m+1)\rho^2 j(\rho) - 2(m+1)\beta^2 j(\rho). \tag{56}$$

ubstituting eqns (46)–(47) into (54) and (56) and substituting (56) into (55) yields xpressions for $h_2(\beta)$ and $h_3(\beta)$. Putting these into eqns (52) and (53) determines $\bar{\psi}_1(y)$ and $_2(y)$. From eqns (37) and (38). $\psi_1(\xi)$ and $\psi_2(\xi)$ may be given as

$$_1(\xi) = \frac{vb\xi}{2\pi(\kappa+1)}[b_r\cos(n\theta_0) - b_\theta\sin(n\theta_0)]J_{n-1}(\xi r_0)$$
$$-\frac{\xi(2-v)}{2\pi(\kappa+1)}[b_r\cos(n\theta_0) + b_\theta\sin(n\theta_0)]J_{n+1}(\xi r_0)$$
$$-\frac{\sqrt{\xi}}{\sqrt{2\pi}}\frac{1}{\pi(\kappa+1)}r_0^{-n+1}\int_0^a \frac{t^{n+1/2}J_{n-1/2}(\xi t)L(t)\,dt}{(r_0^2-t^2)^{3/2}}, \tag{57}$$

$$_2(\xi) = -\frac{1}{b}\psi_1(\xi) + \frac{\xi(2-v)}{2\pi bv^2}[b_r\cos(n\theta_0) + b_\theta\sin(n\theta_0)]J_{n+1}(\xi r_0)$$
$$+\frac{\sqrt{\xi}}{\sqrt{2\pi}}\frac{(2-v)}{b\pi v^2}[b_r\cos(n\theta_0) + b_\theta\sin(n\theta_0)]r_0^{-n-1}\int_0^a \frac{t^{n+5/2}J_{n+1/2}(\xi t)\,dt}{(r_0^2-t^2)^{3/2}}, \tag{58}$$

ith

$$(t) = -vb[b_r\cos(n\theta_0) - b_\theta\sin(n\theta_0)] - (2-v)(2n+1)$$
$$\times[b_r\cos(n\theta_0) + b_\theta\sin(n\theta_0)] + 2n(2-v)[b_r\cos(n\theta_0) + b_\theta\sin(n\theta_0)]\frac{t^2}{r_0^2}. \tag{59}$$

ı deriving eqns (57) and (58), the following divergent integrals have been used to subtract ıe full space contribution

$$\int_0^{r_0} \frac{t^{n+1/2}J_{n-1/2}(\xi t)\,dt}{(r_0^2-t^2)^{3/2}} = -\frac{\sqrt{\pi\xi}}{\sqrt{2}}r_0^{n-1}J_{n-1}(\xi r_0), \tag{60}$$

$$\int_0^{r_0} \frac{t^{n+5/2}J_{n-1/2}(\xi t)\,dt}{(r_0^2-t^2)^{3/2}} = -\frac{\sqrt{\pi\xi}}{2\sqrt{2n}}r_0^{n+1}\{J_{n+1}(\xi r_0) + (2n+1)J_{n-1}(\xi r_0)\}. \tag{61}$$

se of eqns (35) and (36) allows $F_n^s(\xi)$ and $D_n^s(\xi)$ to be evaluated. Substituting the non-tegral terms of eqns (57) and (58) into eqns (35) and (36) verifies that the full space ılution given in eqns (19) and (20) is obtained. The integral terms again represent the teraction between the dislocation and the crack.

It remains to determine the functions $D_0^c(\xi)$, $D_n^c(\xi)$ and $F_n^c(\xi)$. These can be found by ınsidering the sine expansion of Φ and the cosine expansion of ψ [thus using eqns (9)–2)]. The following definitions are made

$$\xi^2 D_0^c(\xi) = \omega_0(\xi), \qquad 2\xi^2 D_n^c(\xi) = -\omega_2(\xi) - \frac{v}{2-v}\omega_1(\xi), \tag{62, 63}$$

$$2\xi^2 F_n^c(\xi) = \omega_2(\xi) - \frac{v}{2-v}\omega_1(\xi). \tag{64}$$

Using the substitutions in eqn (37) as well as

$$\omega_0(y/a) = \bar{\omega}_0(y), \quad \omega_1(y/a) = \bar{\omega}_1(y), \quad \omega_2(y/a) = \bar{\omega}_2(y), \tag{65}$$

the remaining boundary conditions in eqns (32)–(33) become

$$\int_0^\infty y\bar{\omega}_0(y)J_1(y\rho)\,\mathrm{d}y = 0, \quad \rho < 1, \qquad \int_0^\infty \bar{\omega}_0(y)J_1(y\rho)\,\mathrm{d}y = f(\rho), \quad \rho > 1, \tag{66, 67}$$

$$\int_0^\infty y\bar{\omega}_2(y)J_{m+2}(y\rho)\,\mathrm{d}y = 0, \quad \rho < 1, \qquad \int_0^\infty y\bar{\omega}_1(y)J_m(y\rho)\,\mathrm{d}y = 0, \quad \rho < 1, \tag{68, 69}$$

$$\frac{2}{\rho}\int_0^\infty [\bar{\omega}_1(y) + b\bar{\omega}_2(y)]J_{m+2}(y\rho)\,\mathrm{d}y = j(\rho), \quad \rho > 1, \tag{70}$$

$$\frac{2}{\rho}\int_0^\infty [\bar{\omega}_1(y) + \bar{\omega}_2(y)]J_m(y\rho)\,\mathrm{d}y = h(\rho), \quad \rho > 1. \tag{71}$$

The functions $f(\rho)$, $j(\rho)$ and $h(\rho)$ are now defined as

$$f(\rho) = \frac{b_\theta}{2\pi\rho}\delta(a[\rho-\rho_0]), \tag{72}$$

$$j(\rho) = \frac{2-v}{\pi v^2\rho^2}\delta(a[\rho-\rho_0])[b_r \sin(n\theta_0) - b_\theta\cos(n\theta_0)], \tag{73}$$

$$h(\rho) = \frac{1}{\pi v\rho^2}\delta(a[\rho-\rho_0])[b_r \sin(n\theta_0) + b_\theta\cos(n\theta_0)]. \tag{74}$$

The integral equations (66)–(71) are analogous to those in eqns (39)–(44). The solution for $\bar{\omega}_0(y)$ is given in eqn (48) while $\bar{\omega}_1(y)$ and $\bar{\omega}_2(y)$ are given in eqns (52)–(56). Equations (72)–(74) are used for $f(\rho)$, $j(\rho)$ and $h(\rho)$. Omitting the details, the final results are

$$D_0^c(\xi) = \frac{b_\theta}{8\pi\xi}\left[J_1(\xi r_0) + \frac{\sqrt{2}}{\sqrt{\pi\xi}}r_0^{-1}\int_0^a \frac{x^{5/2}}{(r_0^2-x^2)^{3/2}}J_{3/2}(\xi x)\,\mathrm{d}x\right], \tag{75}$$

$$\begin{aligned}\omega_1(\xi) = {} & \frac{vb\xi}{2\pi(\kappa+1)}[b_r\sin(n\theta_0) + b_\theta\cos(n\theta_0)]J_{n-1}(\xi r_0) \\ & - \frac{\xi(2-v)}{2\pi(\kappa+1)}[b_r\sin(n\theta_0) - b_\theta\cos(n\theta_0)]J_{n+1}(\xi r_0) \\ & - \frac{\sqrt{\xi}}{\sqrt{2\pi}}\frac{1}{\pi(\kappa+1)}r_0^{-n+1}\int_0^a \frac{t^{n+1/2}J_{n-1/2}(\xi t)M(t)\,\mathrm{d}t}{(r_0^2-t^2)^{3/2}},\end{aligned} \tag{76}$$

$$\omega_2(\xi) = -\frac{1}{b}\omega_1(\xi)+\frac{\xi(2-\nu)}{2\pi b\nu^2}[b_r\sin(n\theta_0)-b_\theta\cos(n\theta_0)]J_{n+1}(\xi r_0)$$
$$+\frac{\sqrt{\xi}}{\sqrt{2\pi}}\frac{(2-\nu)}{b\pi\nu^2}[b_r\sin(n\theta_0)-b_\theta\cos(n\theta_0)]r_0^{-n-1}\int_0^a\frac{t^{n-5/2}J_{n+3/2}(\xi t)\,\mathrm{d}t}{(r_0^2-t^2)^{3/2}}, \quad (77)$$

with

$$M(t) = -\nu b[b_r\sin(n\theta_0)+b_\theta\cos(n\theta_0)]-(2-\nu)(2n+1)$$
$$\times[b_r\sin(n\theta_0)-b_\theta\cos(n\theta_0)]+2n(2-\nu)[b_r\sin(n\theta_0)-b_\theta\cos(n\theta_0)]\frac{t^2}{r_0^2}. \quad (78)$$

Substituting eqns (76) and (77) into eqns (63) and (64) allows $D_n^c(\xi)$ and $F_n^c(\xi)$ to be evaluated. The full space solution in eqns (24)–(26) are again obtained from the closed form terms in eqns (75)–(77).

The two quantities of interest now are the displacement discontinuities inside the crack and the shear stress on the crack plane outside the crack. The determination of the crack opening displacement requires the evaluation of the expressions in eqns (5), (6), (9) and (10). Adding eqns (5) and (9) gives the total radial displacement $u(r,\theta,0)$ for the half space while superposing eqns (6) and (10) yields the total angular displacement $v(r,\theta,0)$. The first step is to perform the back substitutions for the functions $F_n(\xi)$ and $D_n(\xi)$. Here consideration need only be given to the terms containing integrals on $0\to a$, since the closed form expressions represent the dislocation in the absence of the crack and give a zero contribution. The displacements may be found in closed form by first evaluating the Bessel integrals as

$$\int_0^\infty t^{1+\mu-\lambda}J_\lambda(at)J_\mu(bt)\,\mathrm{d}t = \frac{b^\mu(a^2-b^2)^{\lambda-\mu-1}H(a-b)}{2^{\lambda-\mu-1}a^\lambda\Gamma(\lambda-\mu)}, \quad (79)$$

and then applying the result

$$\int_r^a\frac{t}{(r_0^2-t^2)^{3/2}(t^2-r^2)^{1/2}}\,\mathrm{d}t = \frac{\sqrt{a^2-r^2}}{\sqrt{r_0^2-a^2}}\frac{1}{r_0^2-r^2}. \quad (80)$$

The sums may be evaluated by the formulae in Appendix A. After some rather lengthy manipulations the final results may be given as

$$\pi^2(2-\nu)\frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}u(r,\theta,0) = b_r\left\{\frac{2(1+\nu)\cos(\theta-\theta_0)-\nu(r/r_0)}{2R^2}\right.$$
$$\left.-\frac{2\nu rr_0\sin^2(\theta-\theta_0)}{R^4}\right\}+b_\theta\sin(\theta-\theta_0)\left\{\frac{(1-\nu)}{R^2}+\frac{\nu(r^2-r_0^2)}{R^4}\right\}, \quad (81)$$

$$\pi^2(2-\nu)\frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}v(r,\theta,0) = b_r\sin(\theta-\theta_0)\left\{-\frac{1}{R^2}+\frac{\nu(r^2-r_0^2)}{R^4}\right\}$$
$$+b_\theta\left\{\frac{2(1-2\nu)\cos(\theta-\theta_0)+\nu(r/r_0)}{2R^2}+\frac{2\nu rr_0\sin^2(\theta-\theta_0)}{R^4}\right\}. \quad (82)$$

Multiplying the above expressions by two gives the total displacement discontinuity across the crack faces. The above equations are in agreement with the isotropic limit of the results recently given by Fabrikant (1987b).

To determine the shear stresses, the integral and nonintegral terms of eqns (51), (57) and (58) as well as eqns (75)–(77) are again considered separately. The closed form expressions from these equations represent the dislocation in a full space and hence the shear stress on the surface is given by eqns (30), (31). It remains to determine the contribution from the terms containing integrals on $0 \to a$, which represent the interaction. The resulting expressions [after the back substitutions are made and eqns (7), (8), (11) and (12) are appropriately superposed] may be evaluated as follows. First the Bessel integrals may be performed by using eqn (79). Then the sums can be evaluated by using Appendix A. The resulting expressions may be put in the form

$$\frac{\pi^2(\kappa+1)}{\mu}\tau_{zr}(r,\theta,0) = \frac{\pi^2(\kappa+1)}{\mu}\tau^{\mathrm{F}}_{zr}(r,\theta,0)+\int_0^a \frac{[b_r P(t)+b_\theta Q(t)]\,\mathrm{d}t}{(r_0^2-t^2)^{3/2}(r^2-t^2)^{3/2}}, \tag{83}$$

$$\frac{\pi^2(\kappa+1)}{\mu}\tau_{z\theta}(r,\theta,0) = \frac{\pi^2(\kappa+1)}{\mu}\tau^{\mathrm{F}}_{z\theta}(r,\theta,0)+\int_0^a \frac{[b_r R(t)+b_\theta S(t)]\,\mathrm{d}t}{(r_0^2-t^2)^{3/2}(r^2-t^2)^{3/2}}, \tag{84}$$

where $\tau^{\mathrm{F}}_{zr}(r,\theta,0)$ and $\tau^{\mathrm{F}}_{z\theta}(r,\theta,0)$ are the full space solutions from eqns (30), (31) and the functions $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ are given in Appendix B. From eqns (83), (84) it is evident that the shear stresses contain a square root singularity as $r \to a^+$ or $r_0 \to a^+$. All of the integrals in (83), (84) can be evaluated using the results in Appendix C. The shear stresses then become

$$\begin{aligned}
\frac{\pi^2(\kappa+1)}{\mu}\tau_{zr}(r,\theta,0) &= \frac{\pi^2(\kappa+1)}{\mu}\tau^{\mathrm{F}}_{zr}(r,\theta,0)+\frac{b_r}{(2-\nu)}\left[\frac{2a}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right.\\
&\times\left\{-\frac{\nu}{rr_0}\frac{a^4}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}+\frac{2(1+\nu)\cos(\theta-\theta_0)}{R^2}\right.\\
&\left.-\frac{4\nu rr_0\sin^2(\theta-\theta_0)}{R^4}\right\}+\frac{2}{R^3}\tan^{-1}\left[\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right]\\
&\times\left\{-(2+\nu-\nu^2)\cos(\theta-\theta_0)+\frac{3\nu(2-\nu)rr_0\sin^2(\theta-\theta_0)}{R^2}\right\}\\
&-\frac{2\nu a}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}\left\{(1+\nu)\cos(\theta-\theta_0)\right.\\
&\left.+\frac{(2-3\nu)rr_0\sin^2(\theta-\theta_0)}{R^2}\right\}+\frac{4\nu^2a^3rr_0\sin^2(\theta-\theta_0)}{R^2}\\
&\left.\times\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\right]+\frac{b_\theta}{(2-\nu)}\left[\frac{4a\sin(\theta-\theta_0)}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right.\\
&\times\left\{\frac{(1-\nu)}{R^2}+\frac{\nu[r^2-(1-\nu)r_0^2]}{R^4}-\frac{\nu^2a^2r^2r_0^2}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\right\}\\
&+\frac{2}{R^3}\sin(\theta-\theta_0)\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right)\left\{(\nu^2+2\nu-2)\right.\\
&\left.-\frac{3\nu[r^2-(1-\nu)r_0^2-\nu rr_0\cos(\theta-\theta_0)]}{R^2}\right\}+\frac{2\nu a}{R^2}\sin(\theta-\theta_0)\\
&\times\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}\left\{-\nu+\frac{[r^2-(1-\nu)r_0^2-3\nu rr_0\cos(\theta-\theta_0)]}{R^2}\right\}
\end{aligned}$$

$$-\frac{4\nu^2 a^3 r r_0 \sin(\theta-\theta_0)\cos(\theta-\theta_0)}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\Bigg]. \tag{85}$$

$$\frac{\pi^2(\kappa+1)}{\mu}\tau_{z\theta}(r,\theta,0) = \frac{\pi^2(\kappa+1)}{\mu}\tau^{F}_{z\theta}(r,\theta,0)+\frac{b_r}{(2-\nu)}\Bigg[\frac{4a\sin(\theta-\theta_0)}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}$$

$$\times\left\{-\frac{(1-\nu)}{R^2}-\frac{\nu[r_0^2-(1-\nu)r^2]}{R^4}+\frac{\nu^2a^2r^2r_0^2}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\right\}$$

$$+\frac{2}{R^3}\sin(\theta-\theta_0)\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right)\Bigg\{-(\nu^2+2\nu-2)$$

$$+\frac{3\nu[r_0^2-(1-\nu)r^2-\nu rr_0\cos(\theta-\theta_0)]}{R^2}\Bigg\}+\frac{2\nu a}{R^2}\sin(\theta-\theta_0)$$

$$\times\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}\left\{\nu-\frac{[r_0^2-(1-\nu)r^2-3\nu rr_0\cos(\theta-\theta_0)]}{R^2}\right\}$$

$$+\frac{4\nu^2a^3rr_0\sin(\theta-\theta_0)\cos(\theta-\theta_0)}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\Bigg]$$

$$+\frac{b_\theta}{(2-\nu)}\Bigg[\frac{2a(1-\nu)}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\left\{\frac{\nu}{rr_0}\frac{a^4}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}\right.$$

$$\left.+\frac{2(1-2\nu)\cos(\theta-\theta_0)}{R^2}+\frac{4\nu rr_0\sin^2(\theta-\theta_0)}{R^4}\right\}-\frac{2}{R^3}\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right)$$

$$\times\left\{(2-5\nu+4\nu^2)\cos(\theta-\theta_0)+\frac{3\nu(2-3\nu)rr_0\sin^2(\theta-\theta_0)}{R^2}\right\}$$

$$+\frac{2\nu a}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}\left\{\cos(\theta-\theta_0)+\frac{(2-5\nu)rr_0\sin^2(\theta-\theta_0)}{R^2}\right\}$$

$$-\frac{4\nu^2a^3rr_0\sin^2(\theta-\theta_0)}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}\Bigg]. \tag{86}$$

The modes II and III stress intensity factors are the coefficients of the square root singularities and are defined as

$$K_{\mathrm{II}}(\theta)=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\,\tau_{zr}(r,\theta,0),\qquad K_{\mathrm{III}}(\theta)=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\,\tau_{z\theta}(r,\theta,0). \tag{87, 88}$$

From the previous results it is easy to verify

$$\frac{\pi^2(\kappa+1)(2-\nu)}{2\mu\sqrt{\pi a}}\sqrt{r_0^2-a^2}\,[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]K_{\mathrm{II}}(\theta)=$$

$$b_r\left\{2(1+\nu)\cos(\theta-\theta_0)-\nu\frac{a}{r_0}-\frac{4\nu ar_0\sin^2(\theta-\theta_0)}{[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]}\right\}$$

$$+2b_\theta\sin(\theta-\theta_0)\left\{(1-\nu)+\frac{\nu(a^2-r_0^2)}{[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]}\right\}, \tag{89}$$

$$\frac{\pi^2(2-\nu)}{\mu\sqrt{\pi a}}\sqrt{r_0^2-a^2}\,[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]K_{\mathrm{III}}(\theta) =$$

$$b_r\sin(\theta-\theta_0)\left\{-1+\frac{\nu(a^2-r_0^2)}{[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]}\right\}$$

$$+b_\theta\left\{(1-2\nu)\cos(\theta-\theta_0)+\frac{\nu a}{2r_0}+\frac{2\nu ar_0\sin^2(\theta-\theta_0)}{[a^2+r_0^2-2ar_0\cos(\theta-\theta_0)]}\right\}. \quad (90)$$

The shear stress intensity factors found by Gao (1989) and the shear stresses given by Gao and Rice (1989) for a point coplanar shear dislocation ahead of a half plane crack are obtained in the limit $a\to\infty$.

5. INTEGRAL FORMULATION FOR MULTIPLE OR WAVY CRACKS

An integral formulation for multiple coplanar cracks or cracks with large amplitude variations in the crack front curvature, subjected to shear loading, can be obtained by distributing the point dislocation solution. The derivation of the nonsingular integral equations for shear loading of multiple coplanar circular cracks used by Fabrikant (1989) is straightforward and left to the interested reader. Here new singular integral equations are derived.

Consider first the case of a finite size planar crack with a wavy crack front. Here the total crack domain is designated as S_t while that of a circular region totally encompassed in S_t is denoted as S_c. A polar coordinate system is taken centered in S_c. The crack occupying S_t can be modeled as a penny-shaped crack in S_c with distributed dislocations to account for the area S_t-S_c. The total stress acting on the region S_t must equal the applied loadings $p_r(r,\theta)$ and $p_\theta(r,\theta)$. The boundary conditions in S_c are met by applying the loadings p_r and p_θ to the penny-shaped crack. If the total shear stresses outside of the region S_c caused by a point dislocation at (r_0,θ_0) are denoted as $\tau^1_{rz}(r,\theta,r_0,\theta_0)$ and $\tau^1_{\theta z}(r,\theta,r_0,\theta_0)$ [which are given in eqns (85), (86) with b_r and b_θ replaced with $b_r(r_0,\theta_0)$ and $b_\theta(r_0,\theta_0)$], then the boundary conditions for S_t-S_c become

$$\tau^0_{rz}(r,\theta)+\iint_{S_t-S_c}\tau^1_{rz}(r,\theta,r_0,\theta_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\theta_0 = p_r(r,\theta),\quad r,\theta\in S_t-S_c, \quad (91)$$

$$\tau^0_{\theta z}(r,\theta)+\iint_{S_t-S_c}\tau^1_{\theta z}(r,\theta,r_0,\theta_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\theta_0 = p_\theta(r,\theta),\quad r,\theta\in S_t-S_c. \quad (92)$$

Here $\tau^0_{rz}(r,\theta)$ and $\tau^0_{\theta z}(r,\theta)$ are the known shear stresses outside of S_c when p_r and p_θ are applied to the faces of S_c [see Sankar and Fabrikant (1983) for explicit expressions]. The boundary conditions in S_c remain satisfied since the dislocation solution introduces no additional shear stresses in S_c. Equations (91), (92) are coupled singular integral equations to determine the crack openings $b_r(r_0,\theta_0)$ and $b_\theta(r_0,\theta_0)$ in S_t-S_c. In contrast to the nonsingular integral equations in Fabrikant (1989), these integral equations are strongly (hyper) singular. The singular terms in (91), (92) arise from the contribution of the stresses corresponding to the point dislocation in a full space given in equations (30), (31). Although the hyper-singular terms render the integrals divergent, their contribution to the integral equations (91), (92) are as the finite part of the divergent integrals.

The advantage here, over previous integral formulations, is that only the reduced crack domain S_t-S_c need be considered in a numerical solution. A numerical solution to these equations is not approached presently, however equations of this type have received considerable attention recently. A numerical solution to (91), (92) follows analogously to the procedure given by Murakami (1985) for the surface crack, as well as Lee *et al.* (1987) and Hanson *et al.* (1989) who considered a bimaterial interface. A method for the evaluation of the finite part integrals for a triangular discretization is given by Lee and Keer (1986) while regular quadratures can be used for the interaction terms. From eqns (85), (86), the interaction terms also contain a square root (integrable) singularity as $r_0 \to a^-$ which must be accounted for in an accurate numerical treatment.

The procedure outlined above for the wavy crack front problem can also be extended to analyse multiple coplanar cracks of arbitrary geometry in the vicinity of a circular crack. The procedure consists of distributing the dislocation solution to account for the additional crack openings. Equating the total shear stress to the applied shear loading gives rise to coupled hyper-singular integral equations over the additional crack openings. This eliminates the need to consider the domain of the circular crack in the numerical treatment and will allow a more accurate numerical result.

6. CONCLUSIONS

The current study has provided the theoretical treatment necessary to accurately analyse multiple coplanar cracks, or cracks with large amplitude variations in the crack front curvature, subjected to shear loading. Multiple crack interactions may arise when microcracking occurs in the process zone of a macrocrack. Large amplitude variations in the crack front curvature can occur when a crack front impinges on a region of higher fracture toughness. For either case, in order to predict crack advance an accurate numerical solution is needed to determine stress intensity factors. The present integral formulations will allow a higher accuracy numerical treatment over present solution methods.

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APPENDIX A

$$\sum_{1}^{\infty} t^n \cos(n\alpha) = \frac{t[\cos(\alpha) - t]}{1 + t^2 - 2t\cos(\alpha)}, \qquad \sum_{1}^{\infty} t^n \sin(n\alpha) = \frac{t\sin(\alpha)}{1 + t^2 - 2t\cos(\alpha)}, \tag{A1, A2}$$

$$\sum_{1}^{\infty} (2n+1)t^n \cos(n\alpha) = \frac{3t\cos(\alpha) - t^2}{1 + t^2 - 2t\cos(\alpha)} - \frac{4t^2\sin^2(\alpha)}{[1 + t^2 - 2t\cos(\alpha)]^2}, \tag{A3}$$

$$\sum_{1}^{\infty} (2n+1)t^n \sin(n\alpha) = \frac{3t\sin(\alpha)}{1 + t^2 - 2t\cos(\alpha)} + \frac{4t^2\sin(\alpha)[\cos(\alpha) - t]}{[1 + t^2 - 2t\cos(\alpha)]^2}, \tag{A4}$$

$$\sum_{1}^{\infty} n^2 t^n \cos(n\alpha) = -\frac{t\cos(\alpha)[t^2 - 1]}{[1 + t^2 - 2t\cos(\alpha)]^2} + \frac{4t^2\sin^2(\alpha)[t^2 - 1]}{[1 + t^2 - 2t\cos(\alpha)]^3}, \tag{A5}$$

$$\sum_{1}^{\infty} n^2 t^n \sin(n\alpha) = t\sin(\alpha)\left\{\frac{1 + t^2 + 4t\cos(\alpha)}{[1 + t^2 - 2t\cos(\alpha)]^2} - \frac{8t^2\sin^2(\alpha)}{[1 + t^2 - 2t\cos(\alpha)]^3}\right\}. \tag{A6}$$

APPENDIX B

The functions $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ are given as

$$\begin{aligned}(2-\nu)P(t) = {} & \frac{1}{\gamma(t)}\left\{-\frac{2\nu}{rr_0}t^4[r^2r_0^2 - t^4] + 4(1+\nu)\cos(\theta-\theta_0)t^2[r^2r_0^2 - t^4]\right. \\ & \left. + 4\nu\cos(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2)\right\} + \frac{1}{\gamma^2(t)}\{-8\nu rr_0\sin^2(\theta-\theta_0)t^4[r^2r_0^2 - t^4] \\ & - 8\nu rr_0\sin^2(\theta-\theta_0)t^4(r^2-t^2)(r_0^2-t^2) + 4\nu^2\cos(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2)[r^2r_0^2-t^4]\} \\ & + \frac{1}{\gamma^3(t)}\{-16\nu^2 rr_0\sin^2(\theta-\theta_0)t^4(r^2-t^2)(r_0^2-t^2)[r^2r_0^2-t^4]\},\end{aligned} \tag{B1}$$

$$\begin{aligned}(2-\nu)Q(t) = {} & \frac{1}{\gamma(t)}\{(\kappa+1)\sin(\theta-\theta_0)t^2[r^2r_0^2-t^4]\} + \frac{1}{\gamma^2(t)}\{4\nu\sin(\theta-\theta_0) \\ & \times[r^2t^2 - (1-\nu)r_0^2t^2 - \nu r^2r_0^2]t^2[r^2r_0^2-t^4] - 4\nu^2\sin(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2) \\ & \times[r^2r_0^2 + t^4 + 4t^2rr_0\cos(\theta-\theta_0)]\} + \frac{1}{\gamma^3(t)}\{32\nu^2r^2r_0^2\sin^3(\theta-\theta_0)t^6(r^2-t^2)(r_0^2-t^2)\},\end{aligned} \tag{B2}$$

$$\begin{aligned}(2-\nu)R(t) = {} & \frac{1}{\gamma(t)}\{-(\kappa+1)\sin(\theta-\theta_0)t^2[r^2r_0^2-t^4]\} + \frac{1}{\gamma^2(t)}\{4\nu\sin(\theta-\theta_0) \\ & \times[t^2(r^2-r_0^2) - \nu r^2t^2 + \nu r^2r_0^2]t^2[r^2r_0^2-t^4] + 4\nu^2\sin(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2) \\ & \times[r^2r_0^2 + t^4 + 4t^2rr_0\cos(\theta-\theta_0)]\} + \frac{1}{\gamma^3(t)}\{-32\nu^2r^2r_0^2\sin^3(\theta-\theta_0)t^6(r^2-t^2)(r_0^2-t^2)\},\end{aligned} \tag{B3}$$

$$(2-\nu)S(t) = \frac{1}{\gamma(t)}\left\{\frac{2\nu(1-\nu)}{rr_0}t^4[r^2r_0^2-t^4]+4(1-\nu)(1-2\nu)\cos(\theta-\theta_0)t^2[r^2r_0^2-t^4]\right.$$

$$\left.-4\nu(1-\nu)\cos(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2)\right\}+\frac{1}{\gamma^2(t)}\{8\nu(1-\nu)rr_0\sin^2(\theta-\theta_0)t^4[r^2r_0^2-t^4]$$

$$+8\nu(1-\nu)rr_0\sin^2(\theta-\theta_0)t^4(r^2-t^2)(r_0^2-t^2)+4\nu^2\cos(\theta-\theta_0)t^2(r^2-t^2)(r_0^2-t^2)[r^2r_0^2-t^4]\}$$

$$+\frac{1}{\gamma^3(t)}\{-16\nu^2rr_0\sin^2(\theta-\theta_0)t^4(r^2-t^2)(r_0^2-t^2)[r^2r_0^2-t^4]\}. \tag{B4}$$

where

$$\gamma(t) = r^2r_0^2+t^4-2t^2rr_0\cos(\theta-\theta_0). \tag{B5}$$

APPENDIX C

The integrals used to determine the shear stresses may be evaluated as follows. Equation (9) of Fabrikant (1987b) gives the result

$$\int_0^a \frac{\lambda_1(x)\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}} = \frac{1}{R}\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right). \tag{C1}$$

where

$$\lambda_n(x) = \frac{(r^2r_0^2-x^4)}{[r^2r_0^2+x^4-2x^2rr_0\cos(\theta-\theta_0)]^n}. \tag{C2}$$

and $R^2 = r^2+r_0^2-2rr_0\cos(\theta-\theta_0)$. By performing two successive differentiations of eqn (C1) with respect to θ, one may obtain

$$\int_0^a \frac{x^2\lambda_2(x)\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}} = \frac{1}{2R^3}\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right)-\frac{a}{2R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}, \tag{C3}$$

$$\int_0^a \frac{x^4\lambda_3(x)\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}} = \frac{3}{8R^5}\tan^{-1}\left(\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right)$$

$$-\frac{3a}{8R^4}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}-\frac{a^3}{4R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}. \tag{C4}$$

The following integral is evaluated in Appendix B of Hanson (1990)

$$\int_0^a \frac{x^2\lambda_1(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = -\frac{1}{R^3}\tan^{-1}\left[\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right]+\frac{a}{R^2}\frac{1}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}. \tag{C5}$$

Differentiating (C5) with respect to θ yields

$$\int_0^a \frac{x^4\lambda_2(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = -\frac{3}{2R^5}\tan^{-1}\left[\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right]$$

$$+\frac{a}{2R^4}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}+\frac{a}{R^4}\frac{1}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}. \tag{C6}$$

One of the remaining integrals in eqn (83), (84) with a 3/2 singularity is of the following form

$$\int_0^a \frac{x^4\lambda_1(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}}. \tag{C7}$$

The additional x^2 factor in the numerator [compared to eqn (C5)] renders the integral difficult to evaluate in a closed form. It may be partially evaluated by substituting

$$x^4 = [r^2r_0^2+x^4-2x^2rr_0\cos(\theta-\theta_0)]+2x^2rr_0\cos(\theta-\theta_0)-r^2r_0^2. \tag{C8}$$

Using eqn (C5) and the integral

$$\int_0^a \frac{[r^2r_0^2-x^4]\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = \frac{a}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}. \tag{C9}$$

the integral in (C7) takes the form

$$\int_0^a \frac{x^4\lambda_1(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = \frac{a}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}} + 2rr_0\cos-(\theta-\theta_0)$$

$$\times\left\{-\frac{1}{R^3}\tan^{-1}\left[\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right]+\frac{a}{R^2}\frac{1}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right\}-r^2r_0^2\int_0^a \frac{\lambda_1(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}}. \quad \text{(C10)}$$

This last integral can be reduced to one with square root singularities by using the relation

$$\frac{[r^2r_0^2-x^4]}{(r_0^2-x^2)^{3/2}(r^2-x_2)^{3/2}} = \frac{d}{dx}\left[\frac{x}{\sqrt{r_0^2-x^2}\sqrt{r^2-x^2}}\right], \quad \text{(C11)}$$

and an integration by parts. The final result is

$$\int_0^a \frac{x^4\lambda_1(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = \frac{a^5}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\frac{1}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}$$

$$+\frac{2arr_0\cos(\theta-\theta_0)}{R^2}\frac{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]}-\frac{2rr_0\cos(\theta-\theta_0)}{R^3}$$

$$\times\tan^{-1}\left[\frac{aR}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\right]-4r^2r_0^2\int_0^a \frac{[x^4-x^2rr_0\cos(\theta-\theta_0)]\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}[r^2r_0^2+x^4-2x^2rr_0\cos(\theta-\theta_0)]^2}. \quad \text{(C12)}$$

The last integral in eqn (83), (84) with a 3/2 singularity is of the type

$$\int_0^a \frac{x^2\lambda_2(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}}. \quad \text{(C13)}$$

Using eqn (C11) and an integration by parts gives

$$\int_0^a \frac{x^2\lambda_2(x)\,dx}{(r_0^2-x^2)^{3/2}(r^2-x^2)^{3/2}} = \frac{a^3}{\sqrt{r_0^2-a^2}\sqrt{r^2-a^2}}\frac{1}{[r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)]^2}$$

$$-\int_0^a \frac{2x^2\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}[r^2r_0^2+x^4-2x^2rr_0\cos(\theta-\theta_0)]^2}$$

$$+\int_0^a \frac{8x^2[x^4-x^2rr_0\cos(\theta-\theta_0)]\,dx}{(r_0^2-x^2)^{1/2}(r^2-x^2)^{1/2}[r^2r_0^2+x^4-2x^2rr_0\cos(\theta-\theta_0)]^3}. \quad \text{(C14)}$$

The remaining integrals in (C12) and (C14) have square root singularities. When these terms are combined with the other square root singular terms in eqns (83), (84) that do not fit the forms in (C1), (C3), (C4), the resulting combinations can be put in these forms and evaluated.